Indian Statistical Institute
B.Math. (Hons.) III Year
First Semester Exam 2006-07
Introduction to Nonlinear Dynamics

Time: 3 hrs | Date: 29-11-06
Instructor: C R E Raja

Each question carries 10 Marks and Answer any six questions

1. (a) Determine which of the following matrices are hyperbolic/parabolic/elliptic and justify your answer.

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (b) Suppose A is a linear map on $\mathbb{R}^2$ with only one eigenvalue λ but only one linearly independent eigenvector. Then show that A is conjugate to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

2. (a) Let A be a invertible linear map on $\mathbb{R}^2$ and d be the Euclidean metric on $\mathbb{R}^2$. Show that ρ defined by $\rho(x,\ y) = d(Ax,\ Ay)$ for all $x,\ y \in \mathbb{R}^2$ is equivalent to d.

   (b) Let $I = [\alpha,\ \beta]$ be a bounded interval and $f\colon I \to I$ be a non-decreasing continuous map without fixed points in the open interval $(\alpha,\ \beta)$. Then one end point of I is a fixed point of f and all orbits converge to it except the other end point if it is also a fixed point. If f is invertible, then both end points are fixed and all orbits are postively asymptotic to one end point and negatively asymptotic to the other end point.

3. (a) Let $f\colon S^1 \to S^1$ be a homeomorphism and R_ρ be the rotation on S^1 by irrational number ρ, that is, $R_\rho(e^{2\pi ix}) = e^{2\pi ix}e^{2\pi i\rho}$ for all $e^{2\pi ix} \in S^1$. Suppose there is a continuous map $h\colon S^1 \to S^1$ such that $h \circ f = R_\rho \circ h$. Show that if f is not transitive, that is $\{f^n(x) \mid n \in \mathbb{Z}\}$ is not dense in S^1 for any $x \in S^1$, then h is not a homeomorphism.

   (b) Let F and G be lifts of a continuous map $f : S^1 \to S^1$. Then show that there is an integer n such that $F(x) = G(x) + n$ for all $x \in \mathbb{R}^1$.

   (c) Suppose F is a lift of an orientation-preserving homemorphism of S^1. Show that $F - \mathrm{Id}$ is bounded where Id is the identity map on $\mathbb{R}^1$.

4. Let $f\colon S^1 \to S^1$ be an orientation-preserving homemorphism of the circle.

   (a) Let $x, y \in S^1$. Suppose I is a closed interval in S^1 with endpoints x and $f^m(x)$ with $m \neq 0$ and contains $\{f^n(y) \mid n \geq 1\}$. Then f has a periodic point.

   (b) Show that if f has no periodic point, then $\omega(x) = \omega(y)$ for any $x, y \in S^1$: recall that for any $z \in S^1$, $\omega(z) = \cap_{n\geq 1}\overline{\{f^i(z) \mid i \geq n\}}$.

   (c) Suppose $\omega(x)$ is independent of x and is a perfect set. Then f has no periodic point.

5. (a) Consider the dynamical system on $\mathbb{R}^3$

$$x' = 2y(z-1), \quad y' = -x(z-1) \quad \text{and} \quad z' = z.$$

   Is the origin asymptotically stable? Justify your answer.

   (b) Find a strict Liapunov function for the equilibrium $(0,0)$ and find all the equilibriums of

$$x' = -2x - y^2 \quad \text{and} \quad y' = -y - x^2.$$

6. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ having two complex coordinates (w, z) and consider the linear system

$$w' = 2\pi i w \quad \textit{and} \quad z' = 2\pi\theta i z$$

   where θ is an irrational number.

   (a) Find the flow ϕ_t of the system.

   (b) Find L_ω and L_α of any point of $\mathbb{C}^2$.

7. Let W be an open set in $\mathbb{R}^2$ and $f\colon W \to \mathbb{R}^2$ be a C^1-map. Let ϕ be the flow of the system $x' = f$. Let $x \in W$ and $L_\omega(x)$ be the ω-limit points.

   (a) Suppose $L_\omega(x)$ is compact, non-empty and contains no equilibrium. Then the trajectory of any $y \in L_\omega(x)$ is a closed orbit.

   (b) Suppose γ is a closed orbit contained in a limit set $L_\omega(x)$. Then $d(\phi_t(x), \gamma) \to 0$ as $t \to \infty$.

8. Let f be a C^1-map on $\mathbb{R}^2$ and $H\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function. Suppose $DH(x)f(x) = 0$ for all $x \in \mathbb{R}^2$. Then show that

   (a) H is a first integral of $x' = f(x)$,

   (b) if x belongs to a limit cycle, then $DH(x) = 0$ and

   (c) if x belongs to a compact invariant set on which DH is nerver zero, then x lies on a closed orbit.